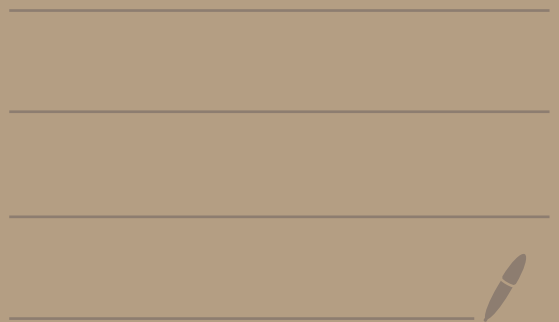


Math 4650

Topic 2a - Application to
calculating square
roots



Application to finding square roots

Theorem: Let $a > 0$ be a real number.

Define the sequence:

$a_1 = \text{any positive real number}$

$$a_{n+1} = \frac{1}{2} \left(a_n + \frac{a}{a_n} \right) \text{ for } n \geq 1$$

Note:

Formula comes from Newton's method

Then:

① $(a_n)_{n=1}^{\infty}$ converges

② $\lim_{n \rightarrow \infty} a_n = \sqrt{a}$

③ $|a_n - \sqrt{a}| \leq \frac{a_n^2 - a}{a_n}$ when $n \geq 2$

error bound

Proof:

① We need two claims first.

Claim (i): $a_n \geq \sqrt{a}$ for $n \geq 2$

Let $n \geq 1$.

By def we have $2a_{n+1} = a_n + \frac{a}{a_n}$.

So, $a_n^2 - 2a_n a_{n+1} + a = 0$.

Thus, $x^2 - 2a_n x + a = 0$ has a real root ($x = a_n$).

So, the discriminant must be non-negative.

That is, $4a_n^2 - 4a \geq 0$.

So, $a_{n+1}^2 \geq a$.

Thus, $a_{n+1} \geq \sqrt{a}$ for $n \geq 1$.

from claim 1:
 $a_n^2 \geq a > 0$
so, $a_n > 0$
and $a_n^2 - a \geq 0$

Claim (ii): $a_n \geq a_{n+1}$ for $n \geq 1$

Let $n \geq 1$.

Then,

$$a_n - a_{n+1} = a_n - \frac{1}{2}a_n - \frac{1}{2}\frac{a}{a_n} = \frac{1}{2}\left(\frac{a_n^2 - a}{a_n}\right) \geq 0$$

Thus, $a_n \geq a_{n+1}$

We have shown that

$$a_2 \geq a_3 \geq a_4 \geq a_5 \geq \dots \geq \sqrt{a} > 0$$

By the monotone convergence theorem,

$(a_n)_{n=1}^{\infty}$ converges.

② Let $L = \lim_{n \rightarrow \infty} a_n$.

We know $a_{n+1} = \frac{1}{2}\left(a_n + \frac{a}{a_n}\right)$ for $n \geq 1$.

Taking the limit of both sides gives

$$L = \frac{1}{2}\left(L + \frac{a}{L}\right)$$

$$\text{So, } 2L^2 = L^2 + a$$

$$\text{Thus, } L^2 = a.$$

And since $L > 0$ we must have $L = \sqrt{a}$.

[We used $a_n > 0 \rightarrow L > 0$. This follows from HW problems]

③ Let $n \geq 2$.

$$\text{Then, } a_n \geq \sqrt{a} \geq \frac{a}{a_n}$$

↑
above

$$\begin{aligned} a_n \geq \sqrt{a} &= \frac{a}{\sqrt{a}} \\ \text{So, } \sqrt{a} &\geq \frac{a}{a_n} \end{aligned}$$

Thus,

$$0 \leq a_n - \sqrt{a} \leq a_n - \frac{a}{a_n} = \frac{a_n^2 - a}{a_n}$$

$$\text{So, } |a_n - \sqrt{a}| \leq \frac{a_n^2 - a}{a_n}$$



Ex: Let's approximate $\sqrt{2}$. Here $a=2$.

Set $a_1 = 1 > 0$.

We have:

$a_{n+1} = \frac{1}{2} \left(a_n + \frac{2}{a_n} \right)$	Error bound $\frac{a_n^2 - a}{a_n}$
$a_2 = \frac{1}{2} \left(1 + \frac{1}{1} \right) = \frac{3}{2} = 1.5$	$\frac{a_2^2 - 2}{a_2} = \frac{1.5^2 - 2}{1.5} \approx 0.1666...$
$a_3 = \frac{1}{2} \left(\frac{3}{2} + \frac{2}{(3/2)} \right) = \frac{17}{12}$ $\approx 1.416666...$	$\frac{a_3^2 - 2}{a_3} = \frac{1}{204} \approx 0.00490196...$
$a_4 = \frac{1}{2} \left(\frac{17}{12} + \frac{2}{17/12} \right) = \frac{577}{408}$ $\approx 1.414215686...$	$\frac{a_4^2 - 2}{a_4} = \frac{1}{235,416}$ $\approx 0.0000042477996...$
$a_5 = \frac{1}{2} \left(\frac{577}{408} + \frac{2}{577/408} \right)$ $= \frac{665857}{470832} \approx 1.41421356237...$	$\frac{a_5^2 - 2}{a_5} = \frac{1}{313,506,783,024}$ $\approx 3.1897 \times 10^{-12}$

We get rapid convergence here.